UNIVERSITY OF CALIFORNIA PUBLICATIONS
IN
MATHEMATICS

Vol. 1, No. 18, pp. 389-400, 1 figure in text December 19, 1923

A SPECIAL QUARTIC CURVE

BY

ELSIE JEANNETTE McFARLAND

UNIVERSITY OF CALIFORNIA PRESS
BERKELEY, CALIFORNIA

UNIVERSITY OF CALIFORNIA PUBLICATIONS

Note.—The University of California Publications are offered in exchange for the publications of learned societies and institutions, universities and libraries. Complete lists of all the publications of the University will be sent upon request. For sample copies, lists of publications or other information, address the Manager of the University of California Press, Berkeley, California, U. S. A. All matter sent in exchange should be addressed to The Exchange Department, University Library, Berkeley, California, U. S. A.

MATHEMATICS.—Derrick N. Lehmer, Editor. Price per volume, $5.00.
Cited as Univ. Calif. Publ. Math.

Vol. 1.
1. On Numbers which Contain no Factors of the Form p(kp+1), by Henry W. Stager. Pp. 1–26. May, 1912 .... $0.50
2. Constructive Theory of the Unicursal Plane Quartic by Synthetic Methods, by Annie Dale Biddle. Pp. 27–54, 31 text-figures. September, 1912 .... .50
3. A Discussion by Synthetic Methods of Two Projective Pencils of Conics, by Baldwin Munger Woods. Pp. 55–85. February, 1913 .... .50
4. A Complete Set of Postulates for the Logic of Classes Expressed in Terms of the Operation "Exception," and a Proof of the Independence of a Set of Postulates due to Del Ré, by B. A. Bernstein. Pp. 87–96. May, 1914 .... .10
5. On a Tabulation of Reduced Binary Quadratic Forms of a Negative Determinant, by Harry N. Wright. Pp. 97–114. June, 1914 .... .20
6. The Abelian Equations of the Tenth Degree Irreducible in a Given Domain of Rationality, by Charles G. P. Kuschke. Pp. 115–162. June, 1914 .... .50
7. Abridged Tables of Hyperbolic Functions, by F. E. Pernot. Pp. 163–169. February, 1915 .... .10
8. A List of Oughtred's Mathematical Symbols, with Historical Notes, by Florian Cajori. Pp. 171–186. February, 1920 .... .25
9. On the History of Gunter's Scale and the Slide Rule during the Seventeenth Century, by Florian Cajori. Pp. 187–209. February, 1920 .... .35
10. On a Birational Transformation Connected with a Pencil of Cubics, by Arthur Robinson Williams. Pp. 211–222. February, 1920 .... .15
11. Classification of Involutory Cubic Space Transformations, by Frank Ray Morris. Pp. 223–240. February, 1920 .... .25
12. A Set of Five Postulates for Boolean Algebras in Terms of the Operation "Exception," by J. S. Taylor. Pp. 241–248. April, 1920 .... .15
13. Flow of Electricity in a Magnetic Field. Four Lectures, by Vito Volterra. Pp. 249–320, 40 figures in text. May, 1920 .... 1.25
14. The Homogeneous Vector Function and Determinants of the P-th Class, by John D. Barter. Pp. 321–343. November, 1920 .... .35
15. Involutory Quartic Transformations in Space of Four Dimensions, by Nina Alderton. Pp. 354–358. November, 1923 .... .25
16. On the Indeterminate Cubic Equation $x^3 + Dy^3 + D^2z^3 - 3Dxyz = 1$, by Clyde Wolfe. Pp. 359–369. December, 1923 .... .25
17. A Study and Classification of Ruled Quartic Surfaces by means of a point-to-line Transformation, by Bing Chin Wong. Pp. 371–387. November, 1923 .... .25
18. A Special Quartic Curve, by Elsie Jeannette McFarland. Pp. 389–400, 1 figure in text. December, 1923 .... .25

PHILOSOPHY.—George P. Adams and J. Loewenberg, Editors. Volumes 1 and 2 are completed (price, $2.00 each); Volume 3 is in progress.

Vol. 1. Studies in Philosophy prepared in Commemoration of the seventieth birthday of Professor George Holmes Howison, November 29, 1904. 262 pages.
1. McGilvary, The Summum Bonum $0.25
2. Mezes, The Essentials of Human Faculty .... .25
3. Stratton, Some Scientific Apologies for Evil .... .15
4. Rieber, Pragmatism and the *a priori* .... .20
5. Bakewell, Latter-Day Flowing-Philosophy .... .20
6. Henderson, Some Problems in Evolution and Education .... .10
7. Burks, Philosophy and Science in the Study of Education .... .15
8. Lovejoy, The Dialectic of Bruno and Spinoza .... .35
9. Stuart, The Logic of Self-Realization .... .30
10. de Laguna, Utility and the Accepted Type .... .20
11. Dunlap, A Theory of the Syllogism .... .10
12. Overstreet, The Basal Principle of Truth-Evaluation .... .25

UNIVERSITY OF CALIFORNIA PUBLICATIONS
IN
MATHEMATICS

Vol. 1, No. 18, pp. 389-400, 1 figure in text December 19, 1923

A SPECIAL QUARTIC CURVE

BY
ELSIE JEANNETTE McFARLAND

The special quartic curve whose equation is

$$F \equiv ax^4 + by^4 + cz^4 + 2fy^2z^2 + 2gx^2z^2 + 2hx^2y^2 = 0 \tag{1}$$

was studied by Hilda Fay Webb, of the University of California, and the results of her investigation are to be found in her thesis for the Master's degree, submitted in May, 1915.

Miss Webb found the quartic to be a non-singular curve with four double tangents passing through each vertex of the fundamental triangle. The six pairs so obtained form a "syzygetic sextuple." She also found that when the general quartic is represented by $UW = V^2$, the complete condition for the reducibility of the general quartic equation to the special form $F = 0$ is as follows:

Two vertices of bitangent pairs of the same set (of six pairs) must coincide; and the conics U, V, and W must cut the polar of the coincident vertices with respect to the contact conic in pairs of an involution.

It is my purpose in this paper to investigate still further the properties of this special quartic curve, and to set up other conditions for the reducibility of the general equation of the fourth degree to the special form $F = 0$.

THE BITANGENTS

The equation of the quartic being given as $ax^4 + by^4 + cz^4 + 2fy^2z^2 + 2gx^2z^2 + 2hx^2y^2 = 0$, the four double tangents through the point (0, 0, 1) will be of the form

$$\begin{aligned} x &= \pm A_1 y \\ x &= \pm A_2 y \end{aligned} \tag{2}$$

where $\pm A_1$ and $\pm A_2$ are the roots of the equation:

$$(g^2 - ac)A^4 + 2(fg - ch)A^2 + (f^2 - bc) = 0 \tag{3}$$

Those through the point (1, 0, 0) are

$$\begin{aligned} y &= \pm B_1 z \\ y &= \pm B_2 z \end{aligned} \tag{4}$$

where $\pm B_1$ and $\pm B_2$ are the roots of the equation

(5) $$(h^2-ab)B^4+2(gh-af)B^2+(g^2-ac)=0$$

Those through the point (0, 1, 0) are

(6) $$z=\pm C_1x$$
$$z=\pm C_2x$$

where $\pm C_1$ and $\pm C_2$ are the roots of the equation

(7) $$(f^2-bc)C^4+2(fh-bg)C^2+(h^2-ab)=0.$$

We see from equations (2), (4), and (6) that each set of four double tangents through a vertex contains two pairs, the lines of which are harmonically divided by the sides of the reference triangle through that vertex. For example, the lines $z=C_1x$ and $z=-C_1x$ are harmonically divided by $x=0$ and $z=0$.

The quartic $F=0$ is transformed into itself by the following collineations:

(a)	(b)	(c)	(d)
$x=x'$	$x=-x'$	$x=\ \ x'$	$x=-x'$
$y=y'$	$y=-y'$	$y=-y'$	$y=\ \ y'$
$z=z'$	$z=\ \ z'$	$z=\ \ z'$	$z=\ \ z'$

If $z=px+qy$ touches $F=0$ twice, then, since (a), (b), (c), and (d) transform F into itself, the three lines $z=-px-qy$, $z=px-qy$ and $z=-px+qy$, obtained by applying (b), (c), and (d) to $z=px+qy$, are likewise double tangents of the quartic.

Similarly, if $z=px+qy$ is an inflexion tangent, $z=-px-qy$, $z=px-qy$, and $z=-px+qy$ are likewise inflexion tangents.

Let us now take as a double tangent the line $z=px+qy$. The equation

(8) $$(a+cp^4+2gp^2)x^4+4(cp^3q+gpq)x^3y$$
$$+6\left(cp^2q^2+\frac{fp^2}{3}+\frac{gq^2}{3}+\frac{h}{3}\right)x^2y^2+4(cpq^3+fpq)xy^3+(b+cq^4+2fq^2)y^4=0,$$ obtained by substituting $px+qy$ for z in (1) must be a perfect square.

If (8) is written: $Ax^4+4Bx^3y+6Cx^2y^2+4Dxy^3+Ey^4=0$, it will be a perfect square if

(9) $$\frac{AC-B^2}{A}=\frac{CE-D^2}{E}, \text{ or } AD^2-EB^2=0$$

and $$\frac{AC-B^2}{A}=\frac{AD-BC}{2B}, \text{ or } 3ABC-2B^3-A^2D=0$$

The resulting conditions on p and q are:

(10) $(ac^2-cg^2)Q^2+(2acf-2fg^2)Q+(cf^2-bc^2)P^2+(2f^2g-2bcg)P+(af^2-bg^2)=0.$

(11) $(ac^2-cg^2)P^2Q+(ag^2-a^2c)Q+(c^2h-cfg)P^3+(3cgh-acf-2fg^2)P^2+(ach+2g^2h-3afg)P+(agh-a^2f)=0$ where $P=p^2$ and $Q=q^2$.

(12) $pq=0$. This condition merely gives us the double tangents through (1, 0, 0) and (0, 1, 0).

Eliminating Q between (10) and (11) we arrive at an equation of the sixth degree in P.

(13) $(c^5h^2-2c^4fgh+ac^4f^2-abc^5+bc^4g^2)P^6+(6ac^3f^2g-8c^3fg^2h-4ac^4fh+6c^4gh^2-2abc^4g+2bc^3g^3)P^5+(2a^2c^3f^2+13c^3g^2h^2+13ac^2f^2g^2+2ac^4h^2-20ac^3fgh-10c^2fg^3h-3abc^3g^2+bc^2g^4+2a^2bc^4)P^4+(8a^2c^2f^2g-4a^2c^3fh-32ac^2fg^2h+8ac^3gh^2+12acf^2g^3-4cfg^4h+12c^2g^3h^2+4a^2bc^3g-4abc^2g^3)P^3+(10a^2cf^2g^2+a^2c^3h^2+4cg^4h^2-10a^2c^2fgh-20acfg^3h+10ac^2g^2h^2+a^3c^2f^2+4af^2g^4+3a^2bc^2g^2-2abcg^4-a^3bc^3)P^2+(2a^3cf^2g-8a^2cfg^2h+2a^2c^2gh^2+4acg^3h^2+4a^2f^2g^3-4afg^4h-2a^3bc^2g+2a^2bcg^3)P+(a^2cg^2h^2+a^3f^2g^2-2a^2fg^3h-a^3bcg^2+a^2bg^4)=0.$

This equation, however, contains the extraneous factor $c^2P^2+2cgP+g^2$. Accordingly the equation in P reduces to

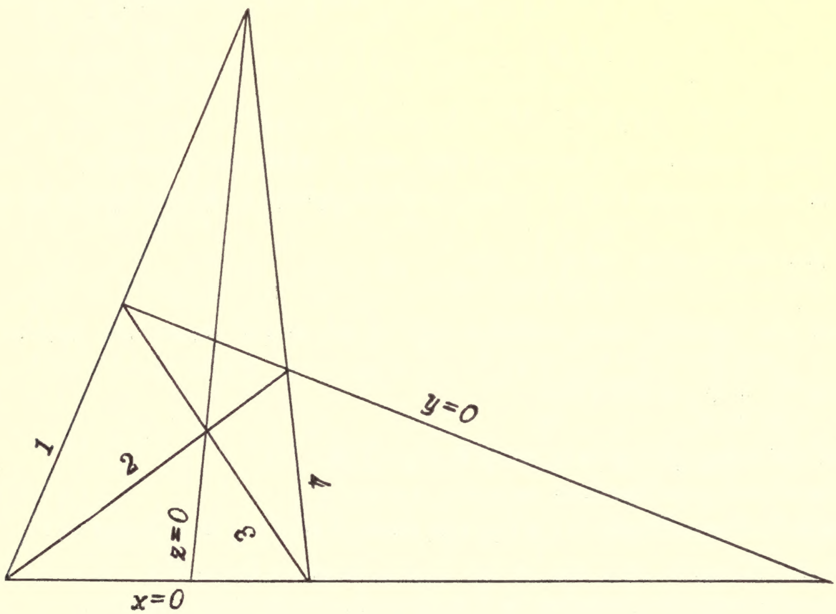
(14) $c^2(ch^2-2fgh+af^2-abc+bg^2)P^4+4c(af^2g-fg^2h-acfh+cgh^2)P^3+2(a^2cf^2+2cg^2h^2-6acfgh+a^2bc^2-abcg^2+2af^2g^2+ac^2h^2)P^2+4a(af^2g-fg^2h-acfh+cgh^2)P+a^2(ch^2-2fgh+af^2-abc+bg^2)=0.$

This equation can, of course, be solved, giving us four values of $P:P_1$, P_2, P_3, and P_4. We have then four corresponding values of $Q:Q_1$, Q_2, Q_3, and Q_4.

Sixteen double tangents are given by the equations:

(15)
$$z=\pm p_1x\pm q_1y \qquad z=\pm p_2x\pm q_2y$$
$$z=\pm p_3x\pm q_3y \qquad z=\pm p_4x\pm q_4y$$
where $p_1=\sqrt{P_1}$ and $q_1=\sqrt{Q_1}$, etc.

Let us represent the four double tangents $z=\pm p_ix\pm q_iy$ by the numbers 1, 2, 3, and 4, and the intersection of any two, say 1 and 2, by (1, 2). Then the points (1, 2) and (3, 4) lie on one side of the reference triangle, points (1, 3) and (2, 4) on another side, and points (1, 4) and (2, 3) on the third side.



Arrangement of the Bitangents in Steiner Complexes

If we pair together the double tangents $x=A_1y$, $x=-A_1y$, and set up the identical relation $F\equiv ax^4+by^4+cz^4+2fy^2z^2+2gx^2z^2+2hx^2y^2$.

(16) $\equiv(x^2-A_1^2y^2)\,(lx^2+my^2+nz^2+rxy+sxz+tyz)-(l'x^2+m'y^2+n'z^2+r'xy+s'xz+t'yz)^2$, we find by equating coefficients and expressing the quantities l, m, n, r, s, t, l', m', n', r', s', t' in terms of l' that we have:

$$(17)\quad \begin{cases} l=a+l'^2 \\ m=\dfrac{-b-\left\{\dfrac{f}{\sqrt{-c}}+\dfrac{A_1^2g}{\sqrt{-c}}+A_1^2l'\right\}^2}{A_1^2} \\ n=2g+2l'\sqrt{-c} \\ r=s=t=0 \end{cases} \qquad \begin{cases} l' \text{ is arbitrary.} \\ m'=-\left\{\dfrac{f}{\sqrt{-c}}+\dfrac{A_1^2g}{\sqrt{-c}}+A_1^2l'\right\} \\ n'=\sqrt{-c} \\ r'=s'=t'=0 \end{cases}$$

We then get

$$(18)\quad F\equiv[x^2-A_1^2y^2]\left[(a+l'^2)x^2-\left\{\frac{b+\left(\dfrac{f+A_1^2g}{\sqrt{-c}}+A_1^2l'\right)^2}{A_1^2}\right\}y^2+(2g+2l'\sqrt{-c})z^2\right.$$

$$-\left[l'x^2-\left(\frac{f+A_1^2g}{\sqrt{-c}}+A_1^2l'\right)y^2+(\sqrt{-c})z^2\right]^2$$ where l' is arbitrary and takes the place of λ as used by Weber throughout his chapter on the double tangents of a quartic.*

Rearranging according to powers of l':

$$(19)\quad \begin{aligned} F\equiv[x^2-A_1^2y^2]&\left[\left\{ax^2-\left(\frac{b+\dfrac{f^2+2fA_1^2g+A_1^4g^2}{-c}}{A_1^2}\right)y^2+2gz^2\right\}\right. \\ &\left.+2l'\left\{-\left(\frac{fA_1^2+gA_1^4}{\sqrt{-c}.A_1^2}\right)y^2+(\sqrt{-c})z^2\right\}+l'^2\{x^2-A_1^2y^2\}\right] \\ &-\left[\left\{-\left(\frac{f+A_1^2g}{\sqrt{-c}}\right).y^2+(\sqrt{-c})z^2\right\}+l'\{x^2-A_1^2y^2\}\right]^2 \\ \equiv[x^2-A_1^2y^2]&[V+2\lambda U+\lambda^2x_1y_1]-[U+\lambda x_1y_1]^2 \end{aligned}$$

We have expressed F in canonical form with l' taking the place of λ.

If the variable conic $V+2\lambda U+\lambda^2x_1y_1\equiv a'x^2+b'y^2+c'z^2+2f'yz+2g'xz+2h'xy=0$, it will degenerate when $\begin{vmatrix} a' & h' & g' \\ h' & b' & f' \\ g' & f' & c' \end{vmatrix}=0.$

Since the variable conic contains only x^2, y^2, and z^2 terms, the determinant becomes $\begin{vmatrix} a' & 0 & 0 \\ 0 & b' & 0 \\ 0 & 0 & c' \end{vmatrix}=a'b'c'=0.$

When $a'=0$ we get the lines $y=\pm B_1z$, $y=\pm B_2z$.

When $b'=0$ we get the lines $z=\pm C_1x$, $z=\pm C_2x$.

When $c'=0$ we get only $x=\pm A_2y$, since c' is of the first degree in l'.

We see, therefore, that the twelve double tangents through the vertices of the fundamental triangle are paired as follows to form a Steiner complex:

* Weber, H., *Lehrbuch der Algebra*, Vol. 2.

$$(20)\quad \begin{array}{lll} (1)\ x=\pm A_1y & (2)\ x=\pm A_2y & (3)\ z=\pm C_1x \\ (4)\ z=\pm C_2x & (5)\ y=\pm B_1z & (6)\ y=\pm B_2z \end{array}$$

If, now, we pair $\begin{matrix} x+A_1y=0 \\ x+A_2y=0 \end{matrix}$ and $\begin{matrix} x-A_1y=0 \\ x-A_2y-0 \end{matrix}$

we shall get a second complex, and if we pair

$$\begin{matrix} x+A_1y=0 \\ x-A_2y=0 \end{matrix} \text{ and } \begin{matrix} x-A_1y=0 \\ x+A_2y=0 \end{matrix}$$

we shall get a third, and these three complexes will contain all twenty-eight double tangents.

Let us now obtain the complex determined by the pair

$$(21)\quad \begin{matrix} x+A_1y=0. \\ x+A_2y=0. \end{matrix} \text{ Let } ax^4+by^4+cz^4+2fy^2z^2+2gx^2z^2+2hx^2y^2\equiv$$
$$[x^2+(A_1+A_2)xy+A_1A_2y^2]\,[lx^2+my^2+nz^2+rxy+sxz+tyz]$$
$$-[l'x^2+m'y^2+n'z^2+r'xy+s'xz+t'yz]^2$$

We find that:

$$l=a+\frac{f^2+2fm'\sqrt{-c}-m'^2c}{-R^2c}+\frac{2fg}{Rc}+\frac{2gm'\sqrt{-c}}{Rc}-\frac{g^2}{c}$$

$$(22)\quad \begin{array}{l|l} m=\dfrac{b+m'^2}{R} & m' \text{ is arbitrary.} \\ n=\dfrac{2f+2m'\sqrt{-c}}{R} & n'=\sqrt{-c} \\ r=\dfrac{2Sfm'}{R^2\sqrt{-c}}+\dfrac{2Sm'^2}{R^2}-\dfrac{Sb}{R^2}-\dfrac{Sm'^2}{R^2} & r'=\dfrac{Sf+Sm'\sqrt{-c}}{R\sqrt{-c}} \\ s=t=0 & s'=t'=0 \\ l^1=\dfrac{f+m'\sqrt{-c}}{R\sqrt{-c}}-\dfrac{g}{\sqrt{-c}} & \end{array}$$

where $\begin{matrix} S=A_1+A_2 \\ R=A_1A_2 \end{matrix}$, if $R=\sqrt{\dfrac{f^2-bc}{g^2-ac}}$

and $S = 2\sqrt{\dfrac{f^2-bc}{g^2-ac}}-\dfrac{2(gf-hc)}{g^2-ac}$, which may be verified from equation (3).

$$(23)\quad \mathrm{F}\equiv[x^2+Sxy+Ry^2]\left[\left\{\left(a-\frac{f^2}{R^2c}+\frac{2fg}{Rc}-\frac{g^2}{c}\right)x^2+\frac{b}{R}y^2+\frac{2f}{R}z^2-\frac{Sb}{R^2}\cdot xy\right\}\right.$$
$$\left.+2m'\left\{\left(\frac{g\sqrt{-c}}{Rc}-\frac{f\sqrt{-c}}{R^2c}\right)x^2+\frac{\sqrt{-c}}{R}z^2+\frac{Sf}{R^2\sqrt{-c}}xy\right\}+m'^2\left\{\frac{x^2}{R^2}+\frac{Ry^2}{R^2}+\frac{Sxy}{R^2}\right\}\right]$$
$$-\left[\left\{\left(\frac{f}{R\sqrt{-c}}-\frac{g}{\sqrt{-c}}\right)x^2+\sqrt{-c}.\ z^2+\frac{Sf}{R\sqrt{-c}}.xy\right\}+m'\left\{\frac{x^2}{R}+y^2+\frac{S}{R}.xy\right\}\right]^2$$

$$(24)\quad \equiv [x^2+Sxy+Ry^2]\left[\left\{\left(a-\frac{f^2}{R^2c}+\frac{2fg}{Rc}-\frac{g^2}{c}\right)x^2+\frac{b}{R}y^2+\frac{2f}{R}z^2-\frac{Sb}{R^2}\cdot xy\right\}\right.$$

$$(24)\qquad +2\lambda\left\{\left(\frac{g\sqrt{-c}}{c}-\frac{f\sqrt{-c}}{Rc}\right)x^2+\sqrt{-c}.\,z^2+\frac{Sf}{R\sqrt{-c}}.xy\right\}+\lambda^2\{x^2+Sxy+Ry^2\}\Big]$$

$$-\left[\left\{\left(\frac{g\sqrt{-c}}{c}-\frac{f\sqrt{-c}}{Rc}\right)x^2+\sqrt{-c}.z^2+\frac{Sf}{R\sqrt{-c}}.xy\right\}+\lambda\{x^2+Sxy+Ry^2\}\right]^2$$

which is in canonical form. $\left(\lambda=\dfrac{m'}{R}\right)$

If the variable conic is to degenerate, the following determinant must vanish:

$$\begin{vmatrix} \left(a-\frac{f^2}{R^2c}+\frac{2fg}{Rc}-\frac{g^2}{c}+\frac{2\lambda g\sqrt{-c}}{c}-\frac{2\lambda f\sqrt{-c}}{Rc}+\lambda^2\right) & \left(\frac{-Sb}{2R^2}+\frac{\lambda Sf}{R\sqrt{-c}}+\frac{\lambda^2 S}{2}\right) & 0 \\ \left(\frac{-Sb}{2R^2}+\frac{\lambda Sf}{R\sqrt{-c}}+\frac{\lambda^2 S}{2}\right) & \left(\frac{b}{R}+\lambda^2 R\right) & 0 \\ 0 & 0 & \frac{2f}{R}+2\lambda\sqrt{-c} \end{vmatrix}=0$$

$$=\left(\frac{2f}{R}+2\lambda\sqrt{-c}\right)\left\{\left(R-\frac{S^2}{4}\right)\lambda^4+\left(\frac{2g\sqrt{-c}}{c}R-\frac{2f\sqrt{-c}}{c}+\frac{S^2f\sqrt{-c}}{Rc}\right)\lambda^3\right.$$

$$(25)\qquad +\left(\frac{b}{R}+aR-\frac{f^2}{Rc}+\frac{2fg}{c}-\frac{g^2R}{c}+\frac{S^2f^2}{R^2c}+\frac{S^2b}{2R^2}\right)\lambda^2$$

$$+\left(\frac{2bg\sqrt{-c}}{Rc}-\frac{2bf\sqrt{-c}}{R^2c}-\frac{S^2bf\sqrt{-c}}{R^3c}\right)\lambda$$

$$\left.+\left(\frac{ab}{R}-\frac{bf^2}{R^3c}+\frac{2bfg}{R^2c}-\frac{bg^2}{Rc}-\frac{S^2b^2}{4R^4}\right)\right\}=0$$

If $\dfrac{2f}{R}+2\lambda\sqrt{-c}=0$ we find that the variable conic becomes $R^2x^2+R^3y^2-SR^2xy=0$, or $(x-A_1y)\,(x-A_2y)=0$.

If the second factor is put equal to zero, we shall get four values of λ for which the variable conic will degenerate.

If we pair $(x+A_1y)$ and $(x-A_2y)$ we replace the quantity $S=A_1+A_2$ by $D=A_1,\ -A_2$ and $R=A_1A_2$ by $-R$. Hence it will be seen at once that the remaining double tangents can be found by replacing S by D and R by $-R$ in the expression for the variable conic and in equation (25).

Since the variable conic is of the form $a'x^2+b'y^2+c'z^2+2h'xy=0$, it can degenerate into $(z\text{-}px\text{-}qy)\,(z+px+qy)=0$, or $(z-px+qy)\,(z+px-qy)=0$.

To show that the complex determined by $\left\{\begin{array}{l}x+A_1y=0\\ x+A_2y=0\end{array}\right.$ contains either four pairs of type $\begin{array}{l}z=\ \ \,px+qy\\ z=-px-qy,\end{array}$ or four pairs of type $\begin{array}{l}z=\ \ \,px-qy\\ z=-px+qy,\end{array}$ let us set up the complex determined by one of the pairs, $\begin{array}{l}z=\ \ \,px+qy\\ z=-px-qy.\end{array}$ The variable conic in this case is again of form $a'x^2+b'y^2+c'z^2+2h'xy=0$, as can be determined by carrying out the process of page seven.

It will degenerate if $c'=0$, or if $a'b'+h'^2=0$.

The coefficient c' is identical with λ^2+c, so that the complex contains two pairs of lines of the form $x=Ay$.

If $a'b'+h'^2=0$, the conic degenerates into

$$(z-px-qy)\ (z+px+qy)=0,$$

where p and q must satisfy conditions (10) and (11). It will be seen then that the same complex is determined by any one of the four pairs

$$\begin{matrix} z=\ \ p_1x+q_1y \\ z=-p_1x-q_1y, \end{matrix}\quad \begin{matrix} z=\ \ p_2x+q_2y \\ z=-p_2x-q_2y, \end{matrix}\quad \begin{matrix} z=\ \ p_3x+q_3y \\ z=-p_3x-q_3y, \end{matrix}\quad \text{or}\quad \begin{matrix} z=\ \ p_4x+q_4y \\ z=-p_4x-q_4y. \end{matrix}$$

Hence one of the two complexes determined by $\begin{matrix} x+A_1y=0 \\ x+A_2y=0 \end{matrix}$ and $\begin{matrix} x+A_1y=0 \\ x-A_2y=0 \end{matrix}$ must contain four pairs of type $\begin{matrix} z=\ \ px+qy \\ z=-px-qy \end{matrix}$ and the other four pairs of type $\begin{matrix} z=\ \ px-qy \\ z=-px+qy. \end{matrix}$

If we pair $\begin{matrix} z+C_1x=0 \\ z+C_2x=0 \end{matrix}$ and set up the identity $ax^4+by^4+cz^4+2fy^2z^2+2gx^2z^2+2hx^2y^2\equiv(z^2-Sxz+Rx^2)\ (lx^2+my^2+nz^2+rxz+syz+txy)\ -\ (l'x^2+m'y^2+n'z^2+r'xz+s'yz+t'xy)^2$, we find that $t=t'=s=s'=0$. Hence the variable conic is of the form $a'x^2+b'y^2+c'z^2+2g'xz=0$. It must then degenerate into a pair of straight lines $(z+px+qy)\ (z+px-qy)=0$, or else into a pair $(z-px+qy)\ (z-px-qy)=0$. The complex then contains the pair $\begin{matrix} z-C_1x=0 \\ z-C_2x=0 \end{matrix}$, and four pairs of the form $\begin{matrix} z=-px-qy \\ z=-px+qy \end{matrix}$ or of the form $\begin{matrix} z=px-qy \\ z=px+qy. \end{matrix}$

The complex determined by $\begin{matrix} z+C_1x=0 \\ z-C_2x=0 \end{matrix}$ will contain the pair $\begin{matrix} z-C_1x=0 \\ z+C_2x=0 \end{matrix}$ and the four pairs of double tangents which do not appear in the preceding complex.

In like manner we can show that of the two complexes determined by $\begin{matrix} y+B_1z=0 \\ y+B_2z=0 \end{matrix}$ and $\begin{matrix} y+B_1z=0 \\ y-B_2z=0 \end{matrix}$, one will contain four pairs of the type $\begin{matrix} z=-px-qy \\ z=\ \ px-qy \end{matrix}$ and the other four of the type $\begin{matrix} z=-px+qy \\ z=\ \ px+qy. \end{matrix}$

Although other groupings may be made in the same manner, those already given are the most striking. They may be summarized as follows:

To the complexes determined by $\begin{matrix} x+A_1y=0 \\ x+A_2y=0 \end{matrix}$ and $\begin{matrix} x+A_1y=0 \\ x-A_2y=0 \end{matrix}$ belong the pairs $\begin{matrix} z=\ \ px+qy \\ z=-px-qy \end{matrix}$ and $\begin{matrix} z=-px+qy \\ z=\ \ px-qy. \end{matrix}$

To the two determined by $\begin{matrix} y+B_1z=0 \\ y+B_2z=0 \end{matrix}$ and $\begin{matrix} y+B_1z=0 \\ y-B_2z=0 \end{matrix}$ belong the pairs $\begin{matrix} z=-px-qy \\ z=\ \ px-qy \end{matrix}$ and $\begin{matrix} z=-px+qy \\ z=\ \ px+qy. \end{matrix}$

To the two determined by $\begin{matrix} z+C_1x=0 \\ z+C_2x=0 \end{matrix}$ and $\begin{matrix} z+C_1x=0 \\ z-C_2x=0 \end{matrix}$ belong the pairs $\begin{matrix} z=-px-qy \\ z=-px+qy \end{matrix}$ and $\begin{matrix} z=px-qy \\ z=px+qy. \end{matrix}$

REDUCTION OF THE GENERAL QUARTIC TO THE SPECIAL FORM $ax^4+by^4+cz^4+2fy^2z^2+2gx^2z^2+2hx^2y^2=0$

In his *Higher Plane Curves,* Salmon refers to this quartic, stating that its equation contains implicitly eleven independent constants. This is shown by replacing x by $lx'+my'+nz'$, y by $l'x'+m'y'+n'z'$, and z by $l''x'+m''y'+n''z'$. F can then be written

$$al^4\left(x'+\frac{m}{l}y'+\frac{n}{l}z'\right)^4+bl'^4\left(x'+\frac{m'}{l'}y'+\frac{n'}{l'}z'\right)^4+cl''^4\left(x'+\frac{m''}{l''}y'+\frac{n''}{l''}z'\right)^4$$
$$+2fl'^2l''^2\left(x'+\frac{m'}{l'}y'+\frac{n'}{l'}z'\right)^2\cdot\left(x'+\frac{m''}{l''}y'+\frac{n''}{l''}z'\right)^2+2gl^2l''^2\left(x'+\frac{m}{l}y'+\frac{n}{l}z'\right)^2\cdot$$
$$\cdot\left(x'+\frac{m''}{l''}y'+\frac{n''}{l''}z'\right)^2+2hl^2l'^2\left(x'+\frac{m}{l}y'+\frac{n}{l}z'\right)^2\cdot\left(x'+\frac{m'}{l'}y'+\frac{n'}{l'}z'\right)^2=0,$$

an expression which evidently contains eleven independent constants. Thus we see that the fourteen independent constants in the equation of the general quartic must be subject to three conditions in order that it may be reducible to the special form $F=0$.

CONDITIONS ON THE COEFFICIENTS OF THE GENERAL QUARTIC

In order that the quartic whose equation contains the full fifteen terms shall be of the type discussed in this paper, its double tangents must be such that four of them pass through some point of the plane, four more through another point, and four more through a third point.

If we represent two different double tangents of the fifteen-term quartic (which we may call Q) by $\begin{matrix} z=p_ix+q_iy \\ z=p_jx+q_jy \end{matrix}$ where $i\neq j$ and where i and j assume values from 1 to 28, the equation of degree $\dfrac{28\text{x}27}{2}$ giving the value of $\dfrac{x}{y}$ for the points of intersection of the twenty-eight double tangents, will have for coefficients the elementary symmetric functions of the various quantities $\dfrac{q_j-q_i}{p_i-p_j}$. But since any interchange of the subscripts of the q's requires a corresponding interchange for the p's, it can be shown that the symmetric functions of $\dfrac{q_j-q_i}{p_i-p_j}$ are also symmetric functions of the quantities p and q; hence that they are rational functions of the coefficients of the quartic Q.

If Q is to be of type F, the equation of degree 14x27 (which may be denoted by $V(x,y)$,) must contain a cubic factor repeated six times. For since four double tangents go through a point, six points of intersection fall together. The same thing occurs for two other sets of four double tangents.

If V has this cubic factor occurring to the sixth degree, its first derivative V' will contain the cubic factor to the fifth degree. Hence V and V' must have a common factor of degree fifteen, which is a perfect fifth power. This can be found

by the rational process of division. When this factor is found its fifth root can be extracted. This fifth root is a cubic which can be solved by radicals. The roots of it give $\frac{x}{y}$ for the three points through each of which pass four double tangents. We can then determine the lines joining these three points, and taking them as sides of a new reference triangle we can find the equation of Q referred to this new triangle. If the transformed equation now contains only even powers, Q was a special quartic of the type under consideration.

If we wish to set up conditions under which Q will be reducible to form F, we must first impose the condition that V and V' shall have a common factor of degree fifteen. This means that when sufficient steps have been taken in the process of finding the greatest common divisior, the remainder of degree fourteen must vanish identically. This implies fifteen conditions, which are not, of course, independent. This common factor of degree fifteen must be a perfect fifth power and conditions for that can be set up. These conditions are not, however, sufficient—or have not been proved so—since the pairs of double tangents through a vertex of a certain triangle must also be harmonically divided by the sides of the triangle through that vertex. Also the remaining sixteen double tangents are related in a special way to the sides of this same triangle. In order to determine the additional conditions we should transform to the new reference triangle, and equate the coefficients of odd powers, if such occur, to zero.

The very large number of conditions obtained in this way must reduce ultimately to three.

GEOMETRICAL CONDITIONS ON THE CURVE

If x_1y_1, x_2y_2, and x_3y_3 are three pairs of double tangents belonging to a Steiner complex, the general quartic can be written in the form $\sqrt{x_1\cdot y_1}+\sqrt{x_2\cdot y_2}+\sqrt{x_3\cdot y_3}=0$. If we take the points of intersection of x_1 with y_1, x_2 with y_2, and x_3 with y_3 for the vertices of our fundamental triangle, we can write the equation of the quartic as: $p\sqrt{(x-ay)\ (x-by)}+q\sqrt{(y-cz)\ (y-dz)}+r\sqrt{(z-fx)\ (z-gx)}=0$, or, in rational form:

$$
\begin{aligned}
(27)\quad & (p^4-2fgp^2r^2+f^2g^2r^4)\,x^4+(q^4-2abp^2q^2+a^2b^2p^4)y^4+(r^4-2cdq^2r^2+c^2d^2q^4)z^4+\\
& (a^2p^4+b^2p^4+4abp^4-2abfgp^2r^2-2fgq^2r^2-2p^2q^2)x^2y^2+(c^2q^4+d^2q^4+4cdq^4-\\
& 2abcdp^2q^2-2abp^2r^2-2q^2r^2)y^2z^2+(f^2r^4+g^2r^4+4fgr^4-2cdfgq^2r^2-2cdp^2q^2-2p^2r^2)x^2z^2\\
& +2(-ap^4-bp^4+afgp^2r^2+bfgp^2r^2)x^3y+2(-cq^4-dq^4+abcp^2q^2+abdp^2q^2)y^3z\\
& +2(-fr^4-gr^4+cdfq^2r^2+cdgq^2r^2)z^3x+2(-a^2bp^4-ab^2p^4+ap^2q^2+bp^2q^2)xy^3\\
& +2(-c^2dq^4-cd^2q^4+cq^2r^2+dq^2r^2)yz^3+2(-f^2gr^4-fg^2r^4+fp^2r^2+gp^2r^2)zx^3+\\
& 2\,(-afp^2r^2-agp^2r^2-bfp^2r^2-bgp^2r^2+cfgq^2r^2+dfgq^2r^2+cp^2q^2+dp^2q^2)x^2yz+\\
& 2\,(-acp^2q^2-adp^2q^2-bcp^2q^2-bdp^2q^2+abfp^2r^2+abgp^2r^2+fq^2r^2+gq^2r^2)xy^2z+\\
& 2\,(-cfq^2r^2-dfq^2r^2-cgq^2r^2-dgq^2r^2+acdp^2q^2+bcdp^2q^2+ap^2r^2+bp^2r^2)xyz^2=0.
\end{aligned}
$$

Let us now set up the conditions that through each vertex of the reference triangle there shall pass a third double tangent to the quartic.

A line through the point (1, 0, 0), of the form $y=mz$, will meet the quartic in four points given by:

(28) $(p^4-2fgp^2r^2+f^2g^2r^4)x^4+2\{(-ap^4-bp^4+afgp^2r^2+bfgp^2r^2)m+(-f^2gr^4-fg^2r^4+fp^2r^2+gp^2r^2)\}x^3z+\{(a^2p^4+b^2p^4+4abp^4-2abfgp^2r^2-2fgq^2r^2-2p^2q^2)m^2+2(-afp^2r^2-agp^2r^2-bfp^2r^2-bgp^2r^2+cfgq^2r^2+dfgq^2r^2+cp^2q^2+dp^2r^2)m+(f^2r^4+g^2r^4+4fgr^4-2cdfgq^2r^2-2cdp^2q^2-2p^2r^2)\}x^2z^2+2\{(-a^2bp^4-ab^2p^4+ap^2q^2+bp^2q^2)m^3+(abfp^2r^2+abgp^2r^2+fq^2r^2+gq^2r^2-acp^2q^2-adp^2q^2-bcp^2q^2-bdp^2q^2)m^2+(acdp^2q^2+bcdp^2q^2+ap^2r^2+bp^2r^2-cfq^2r^2-dfq^2r^2-cgq^2r^2-dgq^2r^2)m+(-fr^4-gr^4+cdfq^2r^2+cdgq^2r^2)\}xz^3+\{(q^4-2abp^2q^2+a^2b^2p^4)m^4+2(-cq^4-dq^4+abcp^2q^2+abdp^2q^2)m^3+(c^2q^4+4cdq^4+d^2q^4-2abp^2r^2-2q^2r^2-2abcdp^2q^2)m^2+2(-c^2dq^4-cd^2q^4+cq^2r^2+dq^2r^2)m+(r^4-2cdq^2r^2+c^2d^2q^4)\}z^4=0$, which may be written:

$$Ax^4+4Bx^3z+6Cx^2z^2+4Dxz^3+Ez^4=0.$$

If the line $y-mz$ is to be a double tangent, the above equation must be a perfect square, which will be the case if

$$(1)\ AD^2-EB^2=0 \text{ and}$$
$$(2)\ 3ABC-2B^3-A^2D+0,$$

or

(1) $m\{m^2-(c+d)m+cd\}\{(f+g)(a+b)(q^2-abp^2)m^2+[(f+g)^2abr^2-(f+g)(a+b)(c+d)q^2+(a+b)^2p^2]m+(f+g)(a+b)(cdq^2-r^2)\}=0.$

(2) $\{m^2-(c+d)m+cd\}\{(a+b)(-fg)m+(f+g)\}=0.$

Since m is an extraneous factor, and $m=c$ or d gives us the original double tangents $y-cz=0$ and $y-dz=0$, we may disregard the first two factors of (1) and the first of (2), and concern ourselves with the two equations:

$(f+g)(a+b)(q^2-abp^2)m^2+\{(f+g)^2abr^2-(f+g)(a+b)(c+d)q^2+(a+b)^2p^2\}m+(f+g)(a+b)(cdq^2-r^2)=0.$
and $-fg(a+b)m+(f+g)=0.$

The eliminant of these equations is:

(29) $E_2\equiv(f+g)^2(q^2-abp^2)+fg\{(f+g)^2abr^2-(f+g)(a+b)(c+d)q^2+(a+b)^2p^2\}+f^2g^2(a+b)^2(f+g)(cdq^2-r^2)=0.$

If we had chosen $x=ly$ or $z=nx$, we should have got two other eliminants:

$E_1\equiv(c+d)^2(p^2-fgr^2)+cd\{(c+d)^2fgq^2-(c+d)(f+g)(a+b)p^2+(f+g)^2r^2\}+c^2d^2(f+g)^2(c+d)(abp^2-q^2)=0.$

$E_3\equiv(a+b)^2(r^2-cdq^2)+ab\{(a+b)^2cdp^2-(a+b)(c+d(f+g)r^2+(c+d)^2q^2\}+a^2b^2(c+d)^2(a+b)(fgr^2-p^2)=0.$

It is evident from an inspection of the three eliminants that they will vanish if $a=-b$, $c=-d$, and $f=-g$. But the condition $a=-b$ is just the condition that the two double tangents through the vertex (0, 0, 1) of the reference triangle shall form a harmonic pair with the axes $x=0$, $y=0$. Similarly, if $c=-d$, the double

tangents through (1, 0, 0) are harmonically divided by $y=0$, $z=0$, and finally if $f=-g$, the double tangents through (0, 1, 0) are harmonically divided by $x=0$, $z=0$.

Hence it will be possible to pass a third double tangent to the quartic through each vertex of the reference triangle (chosen as indicated on page 397), if the two original double tangents through each vertex are harmonic conjugates of the sides of the triangle meeting in that vertex.

Let us now set up the conditions that our general quartic curve as given by equation (27) shall contain only even powers of the variables. The nine conditions are as follows:

$$
(30)\quad
\begin{aligned}
&(1)\ p^2(-ap^2-bp^2+afgr^2+bfgr^2) \equiv p^2(a+b)\,(fgr^2-p^2) = 0\\
&(2)\ q^2(-cq^2-dq^2+abcp^2+abdp^2) \equiv q^2(c+d)\,(abp^2-q^2) = 0\\
&(3)\ r^2(-fr^2-gr^2+cdfq^2+cdgq^2) \equiv r^2(f+g)\,(cdq^2-r^2) = 0\\
&(4)\ p^2(-a^2bp^2-ab^2p^2+aq^2+bq^2) \equiv p^2(a+b)\,(q^2-abp^2) = 0\\
&(5)\ q^2(-c^2dq^2-cd^2q^2+cr^2+dr^2) \equiv q^2(c+d)\,(r^2-cdq^2) = 0\\
&(6)\ r^2(-f^2gr^2-fg^2r^2+fp^2+gp^2) \equiv r^2(f+g)\,(p^2-fgr^2)=0\\
&(7)\ (-afp^2r^2-agp^2r^2-bfp^2r^2-bgp^2r^2+cfgq^2r^2+dfgq^2r^2+cp^2q^2+dp^2q^2)\\
&\qquad\qquad \equiv -p^2r^2(a+b)\,(f+g)+(c+d)\,(p^2+fgr^2)q^2=0\\
&(8)\ (-acp^2q^2-adp^2q^2-bcp^2q^2-bdp^2q^2+abfp^2r^2+abgp^2r^2+fq^2r^2+fq^2r^2+gq^2r^2)\\
&\qquad\qquad \equiv -p^2q^2(a+b)\,(c+d)+(f+g)\,(q^2+abp^2)r^2=0\\
&(9)\ (-cfq^2r^2-dfq^2r^2-cgq^2r^2-dgq^2r^2+acdp^2q^2+bcdp^2q^2+ap^2r^2+bp^2r^2)\\
&\qquad\qquad \equiv -q^2r^2(c+d)\,(f+g)+(a+b)\,(r^2+cdq^2)p^2=0
\end{aligned}
$$

From inspection we see that the nine equations are satisfied if $a+b=0$, $c+d=0$, $f+g=0$, or if $p=q=r=0$, and in no other way. But we must reject the system $p=q=r=0$, since if these relations were true we should have no quartic at all. Therefore the nine conditions for the reduction of the general quartic to the special form we are studying are equivalent to three, $a+b=0$, $c+d=0$, $f+g=0$. But these conditions, as we have previously noted, are the conditions that each pair of double tangents through a vertex of the reference triangle shall be harmonically divided by the sides of the reference triangle through that vertex.

Let us first select as reference triangle one whose vertices are the intersections of x_1 with y_1, x_2 with y_2, x_3 with y_3, where x_1y_1, x_2y_2, x_3y_3 are three pairs of double tangents of the general quartic belonging to a Steiner complex. If, then, we can draw through each vertex a third double tangent to the quartic, or if—an equivalent condition—we can show that the two double tangents through each vertex are harmonic conjugates of the sides of the reference triangle through that vertex, the given quartic is reducible to the form $ax^4+by^4+cz^4+2fy^2z^2+2gx^2z^2+2hx^2y^2=0$.

NOTE.—We can express a, b, c, d, f, and g in terms of the moduli of the class of curves, and so obtain the conditions they must satisfy if the quartic is of this variety. Riemann, "*Zur Theorie* der Abelschen Functionen für den Fall p=3, *Gesammelte Werke*, p. 456.

SUMMARY

The quartic curve whose equation is

$$ax^4+by^4+cz^4+2fy^2z^2+2gx^2z^2+2hx^2y^2=0$$

has the following properties:

Four double tangents pass through each vertex of the reference triangle, the four consisting of two pairs which are harmonically divided by the sides of the triangle. The remaining sixteen are grouped by fours, each four consisting of the lines $z=\pm px\pm qy$. The quadrilateral formed by any one of these four-groups has for diagonals the sides of the fundamental triangle of reference.

The actual equations of the double tangents can be determined by the solution of a biquadratic equation.

The six pairs of double tangents through the vertices of the reference triangle belong to a Steiner complex. To the complexes determined by $\begin{matrix} x+A_1y=0 \\ x+A_2y=0 \end{matrix}$ and $\begin{matrix} x+A_1y=0 \\ x-A_2y=0 \end{matrix}$ belong the pairs $\begin{matrix} z+px+qy=0 \\ z-px-qy=0 \end{matrix}$ and $\begin{matrix} z+px-qy=0 \\ z-px+qy=0. \end{matrix}$ To those determined by $\begin{matrix} z-C_1x=0 \\ z-C_2x=0 \end{matrix}$ and $\begin{matrix} z-C_1x=0 \\ z+C_2x=0 \end{matrix}$ belong the pairs $\begin{matrix} z+px+qy=0 \\ z+px-qy=0 \end{matrix}$ and $\begin{matrix} z-px+qy=0 \\ z-px-qy=0. \end{matrix}$ And finally to the two determined by $\begin{matrix} y-B_1z=0 \\ y-B_2z=0 \end{matrix}$ and $\begin{matrix} y-B_1z=0 \\ y+B_2z=0 \end{matrix}$ belong the pairs $\begin{matrix} z+px+qy=0 \\ z-px+qy=0 \end{matrix}$ and $\begin{matrix} z+px-qy-0 \\ z-px-qy=0. \end{matrix}$

Under certain conditions the general quartic can be made to reduce to the special form $F=0$. Let three pairs of double tangents be selected from a Steiner complex and let their intersections be taken as the vertices of a new reference triangle. Then if it is possible to pass a third double tangent through each vertex, or if the two original double tangents are harmonically divided by the sides of the reference triangle through that vertex, the general quartic equation is reducible to the form considered in this paper.

Vol. 2. 1909–1921.

1.	Overstreet, The Dialectic of Plotinus	.25
2.	Hocking, Two Extensions of the Use of Graphs in Formal Logic. 20 figures in text	.15
3.	Hocking, On the Law of History	.20
4.	Adams, The Mystical Element in Hegel's Early Theological Writings	.35
5.	Parker, The Metaphysics of Historical Knowledge	.85
6.	Boas, An Analysis of Certain Theories of Truth	1.00

Vol. 3. 1918–

1.	Rieber, Footnotes to Formal Logic. Pp. 1–177. Cloth, \$2.00; paper, \$1.50. Carriage extra. Weight, cloth, 2 lbs.; paper, 1½ lbs.	
2.	Prall, A Study in the Theory of Value. Pp. 197–290	1.00

INDEPENDENT VOLUMES*

Bacon, Leonard, and Rose, Robert S.	The Lay of the Cid. xiv + 130 pages. Cloth, \$1.35. Carriage extra. Weight, 1½ lbs.
Daniel, J. Frank	The Elasmobranch Fishes. xi + 334 pages, 260 illustrations. Cloth, \$5.50. Carriage extra. Weight, 2 lbs., 10 oz.
Glover, T. R.	Herodotus. (*In press.*)
Grinnell, Joseph Bryant, Harold C. Storer, Tracy I.	The Game Birds of California. x + 642 pages, 16 colored plates, 94 line drawings. Cloth, \$6.00. Carriage extra. Weight, 4½ lbs.
Hart, Walter M.	Kipling the Story Writer. 225 pages. Cloth, \$2.25. Carriage extra. Weight, 1½ lbs.
Lewis, C. I.	Survey of Symbolic Logic. vi + 407 pages. Cloth, \$3.00. Carriage extra. Weight, 2¾ lbs.
McCormac, Eugene I.	James K. Polk: A Political Biography. x + 746 pages, 2 plates. Cloth, \$6.00 Carriage extra. Weight, 3 lbs., 10 oz.
McEwen, George F.	Ocean Temperatures: Their Relation to Solar Radiation and Oceanic Circulation. 130 pages. Paper, \$1.50.
Morley, S. Griswold	Anthero de Quental: Sonnets and Poems. xxiv + 133 pages. Quarter-vellum, \$2.25.
Moses, Bernard	Spain's Declining Power in South America. xx + 440 pages. Cloth, \$4.00. Carriage extra. Weight, 2 lbs., 4 oz.
Petersson, Torsten	Cicero: A Biography. 699 pages. Cloth, \$5.00. Carriage extra. Weight, 4 lbs.
Scott, John A.	The Unity of Homer. 275 pages. Cloth, \$3.25. Carriage extra. Weight, 1 lb.
Slate, Frederick	The Fundamental Equations of Dynamics. xii + 225 pages. Cloth, \$2.00. Carriage extra. Weight, 1½ lbs.
Tolman, Richard C.	Theory of the Relativity of Motion. ix + 242 pages. Cloth, \$1.75. Carriage extra. Weight, 1¾ lbs.

* The works listed as independent volumes are also on sale at The Baker and Taylor Co., 354 Fourth Avenue, New York City.